CCA Math Bonanza

March 24, 2024

Individual Round

I1) Twelve pigeons can eat 28 slices of bread in 5 minutes. Find the number of slices of bread 5 pigeons can eat in 48 minutes.

I2) Let $S(x) = x + 1$ and $V(x) = x^2 - 1$. Find the sum of the squares of all real solutions to $S(V(S(V(x)))) = 1$.

I3) Find the units digit of $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, where there are 2024 2s.

I4) Let x and y be positive integers that are at least 2. Suppose Johnny hits 1 out of every 10 free throws, Abigail hits 1 out of every x free throws, and Demar hits 2 out of every y free throws. It turns out that the mean of Abigail's and Demar's individual free throw percentages are the same as Johnny's free throw percentage. Find the sum of all possible values of x.

I5) Triangle ABC has points D,E,F on segment BC in that order, where D is between B and E, and AD and AE trisect angle BAF. If $\angle BAF = 60°$, $\frac{EF}{EC} = \frac{2}{3}$, and $\frac{AE}{AC} = 2$, find $\angle BAC$.

I6) Byan is playing raven, raven, falcon with his three friends. His friends sit down in a circle, and Byan repeatedly walks clockwise around them, tapping each friend he passes on the head and then saying 'raven' or 'falcon', each with probability $\frac{1}{2}$. The game ends after Byan has said 'falcon' twice. The probability one of his friends will be called a falcon twice can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

I7) An infinite geometric sequence $a_1, a_2, a_3, \dots$ satisfies $a_1 = 1$ and

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_4} \cdots = \frac{1}{2}.$$

The sum of all possible values of a_2 can be expressed as $m + \sqrt{n}$, where m and n are integers and n is not a positive perfect square. Find $100m + n$.

I8) Each vertex of a regular heptagon (7-gon) is colored either red or blue. Find the number of distinct colorings such that no three consecutive vertices have the same color. Two colorings are considered distinct if one cannot be obtained from the other by a rotation of the heptagon.

I9) Find the median value of m over all integers m where $|m^2 + 8m - 65|$ is a perfect power. A perfect power is any integer at least 2 which can be written as a^b, where a, b are integers and $b \geq 2$.

I10) Let ABC be a triangle with side lengths $AB = 7$, $BC = 8$, and $CA = 9$. Let O be the circumcenter of $\triangle ABC$, and let AO, BO, CO intersect the circumcircle of $\triangle ABC$ again at D, E, and F, respectively. The area of convex hexagon $AFBDCE$ can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is square-free. Find $m + n$.

I11) The value of the expression

$$\sum_{n=2}^{\infty} \frac{\binom{n}{2}}{7^{n-2}} = 1 + \frac{3}{7} + \frac{6}{49} + \frac{10}{343} + \frac{15}{2401} + \cdots + \frac{\binom{n}{2}}{7^{n-2}} + \cdots$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

I12) Given that

$$a^2 + b^2 + c^2 + d^2 + \frac{5}{4} = e + \sqrt{a + b + c + d - e},$$

the value of $a+b+c+d+e$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

I13) Call a sequence $a_0, a_1, a_2, \ldots$ of positive integers defined by $a_k = 25a_{k-1} + 96$ for all $k > 0$ a *valid* sequence. Call the *goodness* of a *valid* sequence the maximum value of $\gcd(a_k, a_{k+2024})$ over all k. Call a *valid* sequence *best* if it has the maximal *goodness* across all possible *valid* sequences. Find the second largest a_0 across all *best* sequences.

I14) Larry initially has a one character string that is either 'a', 'b', 'c', or 'd'. Every minute, he chooses a character in the string and:

- if it's an 'a' he can replace it with 'ac' or 'da',
- if it's a 'b' he can replace it with 'cb' or 'bd',
- if it's a 'c' he can replace it with 'cc' or 'ba',
- if it's a 'd' he can replace it with 'dd' or 'ab'.

Larry does the above process for 10 minutes. Find the number of possible strings he can end up with that are a permutation of 'aabbccccddd'.

I15) Let ABC be a triangle with side lengths $AB = 13$, $BC = 15$, $CA = 14$. Let ℓ be the line passing through A parallel to BC. Define H as the orthocenter of $\triangle ABC$, and extend BH to intersect AC at E and ℓ at G. Similarly, extend CH to intersect AB at F and ℓ at D. Let M be the midpoint of BC, and let AM intersect the circumcircle of AEF again at P. The ratio $\frac{PD}{PG}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.